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The Degasperis–Procesi equation with self-consistent sources

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Abstract

The Degasperis–Procesi equation with self-consistent sources (DPESCS) is derived. The Lax representation of the DPESCS is presented. The conservation laws for DPESCS are constructed. The peakon solution of DPESCS is obtained by using the method of variation of constants.

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1. Introduction

Soliton equations with self-consistent sources (SESCS) have attracted much attention in recent years. They are important integrable models in many fields of physics, such as hydrodynamics, state physics, plasma physics, etc [1–18]. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves [5]. The KP equation with self-consistent sources describes the interaction of a long-wave with a short-wave packet propagating on the x–y plane at some angle to each other [2]. The nonlinear Schrödinger equation with self-consistent sources represents the nonlinear interaction of an electrostatic high-frequency wave with the ion acoustic wave in a two-component homogeneous plasma [6]. The constrained flows of soliton hierarchy may be regarded as the stationary systems of the corresponding integrable hierarchy with self-consistent sources [15–18]. Since the Lax representation for constrained flows can be deduced from the adjoint representation of the Lax representation of soliton equation [14], there is a natural way of finding the zero-curvature representation for SESCS [15–18]. From this observation, the soliton equations with self-consistent sources may be viewed as integrable generalizations of the original soliton equations. A systematic way to construct the soliton equations with self-consistent sources is proposed in [15–18].

The Camassa–Holm equation, which was implicitly contained in the class of multihamiltonian systems introduced by Fuchssteiner and Fokas in [20] and explicitly derived as a shallow water wave equation by Camassa and Holm in [21], has the form

$$u_t + 2wu_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.1}$$

Since the works of Camassa and Holm, this equation has become a well-known example of integrable systems and has been studied from many different points of view.

It is a natural question to ask whether there are other third-order dispersive PDEs sharing the integrability properties of the Camassa–Holm equation. An answer has been given in [19] where the method of asymptotic integrability was applied to a family of third-order dispersive PDE conservation laws

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx}\right)_x, \tag{1.2}$$

where $\alpha, c_0, c_1, c_2, c_3$ and γ are some arbitrary constants. Only three equations in this family satisfy the asymptotic integrability conditions. They are the KdV equation, the Camassa–Holm equation and the following new equation:

$$u_t + u_x + 6uu_x + u_{xxx} - \alpha^2\left(u_{xxt} + \tfrac{9}{2}u_x u_{xx} + \tfrac{3}{2}uu_{xxx}\right) = 0. \tag{1.3}$$

By a coordinate transformation, this new equation could be written as [22]

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.4}$$

This encourages us to study the family of equations [22]

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}. \tag{1.5}$$

All the equations in this family have peakon solutions of the form

$$u = \sum_{j=1}^{N} p_j(t)\,\mathrm{e}^{-|x-q_j(t)|}, \tag{1.6}$$

where p_j and q_j satisfy the dynamical system

$$\dot{p}_j = -(b-1)\frac{\partial G_N}{\partial q_j}, \qquad \dot{q}_j = \frac{\partial G_N}{\partial p_j}, \tag{1.7}$$

the generating function G_N reading

$$G_N = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k\,\mathrm{e}^{-|q_j-q_k|}. \tag{1.8}$$

Let $m = u - u_{xx}$. Equation (1.4) could be written as

$$m_t + m_x u + 3mu_x = 0, \tag{1.9}$$

which is called the Degasperis–Procesi (DP) equation. It is shown in [23] that both the Camassa–Holm and the DP equation are derived as members of a one-parameter family of asymptotic shallow water approximations to the Euler equations. They describe the unidirectional propagation of nonlinear shallow-water waves. The DP equation has a third-order Lax pair and a bi-Hamiltonian structure. The existence of its global solutions is considered in [24]. A new integrable hierarchy was extended from the DP equation in [25]. In [26], the N-peakon solutions of the DP equation have been derived. The N-soliton solutions of the DP equation are obtained in [27]. Interesting results on the solutions of this equation have also been obtained in [28–32].

The soliton equation with self-consistent sources was first studied by Melnikov in [1–3]. The problem of finding soliton solutions or other specific solutions for equations with self-consistent sources has been considered in the past by several authors [4–18]. The present paper falls in that line of research, aiming at deriving the Degasperis–Procesi equation with self-consistent sources (DPESCS) and at finding its special explicit solutions.

We first construct the high-order constrained flow of the DP equation. Based on it we establish the DPESCS by regarding the constrained flow of DP equation as the stationary equation of DPESCS in the same way as in [15–18]. The Lax pair of the DPESCS is obtained, which means that the DPESCS is Lax integrable. In [33, 34], we pointed out that soliton equations with self-consistent sources can be regarded as soliton equations with non-homogeneous terms, and accordingly proposed to look for explicit solutions by using the methods of variation of constants. Applying this technique to DPESCS we have been able to find its peakon solutions and peakon–antipeakon solutions.

This paper is organized as follows. In section 2, we extend the DP equation including self-consistent sources and construct its Lax representation. In section 3 we derive its conservation laws. In section 4, the peakon and the peakon–antipeakon solutions are obtained. In section 5, we mention some open problems.

2. The DPESCS and its Lax pair

2.1. The DPESCS

First we construct the high-order constrained flows of the DP equation, then establish the DPESCS and describe how to derive the Lax representation for the DPESCS.

It is known that the Lax pair for the DP equation (1.9) is [22]

$$\psi_{xxx} = \psi_x - m\lambda\psi, \tag{2.1a}$$

$$\psi_t = -\frac{1}{\lambda}\psi_{xx} - u\psi_x + \left(u_x + \frac{2}{3\lambda}\right)\psi. \tag{2.1b}$$

Consider the following equations obtained from the spectral problem and its formal adjoint problem for n distinct real λ_j:

$$q_{j,xxx} = q_{j,x} - m\lambda_j q_j, \qquad j = 1, \ldots, n, \tag{2.2a}$$

$$r_{j,xxx} = r_{j,x} + m\lambda_j r_j, \qquad j = 1, \ldots, n. \tag{2.2b}$$

It is not difficult to find that

$$\frac{\delta\lambda_j}{\delta m} = -\lambda_j q_j r_j, \qquad j = 1, \ldots, n. \tag{2.3}$$

It is known that the DP equation possesses a bi-hamiltonian structure [22], namely:

$$m_t = B_1\frac{\delta H_1}{\delta m} = B_0\frac{\delta H_0}{\delta m}, \tag{2.4}$$

where

$$B_0 = m^{2/3}\partial_x m^{1/3}\left(\partial_x - \partial_x^3\right)^{-1} m^{1/3}\partial_x m^{2/3}, \tag{2.5}$$

$$B_1 = \partial_x\left(1 - \partial_x^2\right)\left(4 - \partial_x^2\right), \tag{2.6}$$

$$H_0 = -\frac{2}{9}\int m\,\mathrm{d}x, \tag{2.7}$$

$$H_1 = -\frac{1}{6}\int u^3\,\mathrm{d}x. \tag{2.8}$$

The high-order constrained flow of the DP equation is obtained from (2.2) for n distinct λ_j, requiring that the 'potential' m obeys the following constraint,

$$B_1\left(\frac{\delta H_1}{\delta m} - \sum_{j=1}^{n}\alpha_j\frac{\delta\lambda_j}{\delta m}\right) = 0, \tag{2.9a}$$

$$q_{j,xxx} = q_{j,x} - m\lambda_j q_j, \tag{2.9b}$$

$$r_{j,xxx} = r_{j,x} + m\lambda_j r_j \qquad j = 1,\ldots,n, \tag{2.9c}$$

where α_j, $j = 1,\ldots,n$, are arbitrary constants.

According to the approach proposed in [15–18], the DPESCS consists of the following equation:

$$m_t = B_1\left(\frac{\delta H_1}{\delta m} - \sum_{j=1}^{n}\alpha_j\frac{\delta\lambda_j}{\delta m}\right)$$

and equations (2.2), which by using (2.3) and taking $\alpha_j = -\frac{1}{6}$ leads to the DPESCS

$$m_t = -um_x - 3u_x m - \frac{1}{6}\sum_{j=1}^{n}\partial(1-\partial^2)(4-\partial^2)(\lambda_j q_j r_j), \tag{2.10a}$$

$$q_{j,xxx} = q_{j,x} - m\lambda_j q_j, \tag{2.10b}$$

$$r_{j,xxx} = r_{j,x} + m\lambda_j r_j, \qquad j = 1,\ldots,n. \tag{2.10c}$$

2.2. *The Lax representation of the DPESCS*

Comparing the DPESCS to the DP equation, we may assume that the Lax presentation of the DPESCS has the form

$$\psi_{xxx} = \psi_x - m\lambda\psi, \tag{2.11a}$$

$$\psi_t = -\frac{1}{\lambda}\psi_{xx} - u\psi_x + \left(u_x + \frac{2}{3\lambda}\right)\psi + a\psi + b\psi_x + c\psi_{xx}, \tag{2.11b}$$

where a, b and c are some functions of q_j and r_j to be determined. Requiring that under (2.10b) and (2.10c) the compatibility condition of (2.11a) and (2.11b), namely $\psi_{xxxt} = \psi_{txxx}$, leads to DPESCS (2.10$a$) enables us to find that

$$a_x - a_{xxx} + 3b_x m\lambda + bm_x\lambda + 3c_{xx}m\lambda + 3c_x m_x\lambda + cm_{xx}\lambda$$
$$= \lambda\sum_{j=1}^{n}\alpha_j\partial(1-\partial^2)(4-\partial^2)(q_j r_j), \tag{2.12a}$$

$$-3a_{xx} - b_{xxx} - 2b_x - 3c_{xx} + 3c_x m\lambda + 2cm_x\lambda = 0, \tag{2.12b}$$

$$3a_x + 3b_{xx} + 2c_x + c_{xxx} = 0. \tag{2.12c}$$

From (2.10*b*) and (2.10*c*) we obtain two identities

$$\partial(1-\partial^2)(4-\partial^2)(q_jr_j) = -3\lambda_j(m_xW(q_j,r_j)+3mW(q_j,r_j)_x), \tag{2.13a}$$

$$W(q_j,r_j)_{xxx} - W(q_j,r_j)_x = \lambda_j(2m_xq_jr_j + 3m(q_jr_j)_x), \tag{2.13b}$$

where $W(q_j,r_j) = q_jr_{j,x} - q_{j,x}r_j$ is the usual Wronskian determinant.

From (2.12*c*) we obtain that

$$3a + 3b_x + 2c + c_{xx} = 0. \tag{2.14}$$

Then (2.14) together with (2.12*b*) leads to

$$a_{xxx} - a_x = \frac{1}{6}\partial(1-\partial^2)(4-\partial^2)c + \left(cm_x\lambda + \frac{3}{2}c_xm\lambda\right)_x, \tag{2.15a}$$

$$2\left(\left(b+\frac{1}{2}c_x\right)_x - \left(b+\frac{1}{2}c_x\right)_{xxx}\right) = 2cm_x\lambda + 3c_xm\lambda. \tag{2.15b}$$

With the relations above we can rewrite (2.12*a*) as

$$-\frac{1}{6}\partial(1-\partial^2)(4-\partial^2)c + \lambda\left(3m\left(b+\frac{1}{2}c_x\right)_x + m_x\left(b+\frac{1}{2}c_x\right)\right)$$
$$= \lambda\sum_{j=1}^{n}\alpha_j\partial(1-\partial^2)(4-\partial^2)(q_jr_j). \tag{2.16}$$

Here we suggest that

$$c = \sum_{j=1}^{n}A_jq_jr_j, \tag{2.17a}$$

$$b + \frac{1}{2}c_x = \sum_{j=1}^{n}B_jW(q_j,r_j), \tag{2.17b}$$

where $A_j, B_j, j = 1,\ldots,n$, are some undetermined constants.

Finally with some calculations to determine A_j and B_j we solve (2.12) for a, b, c yielding

$$a = \sum_{j=1}^{n}\frac{1}{6}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(3\lambda(q_jr_{j,xx} - q_{j,xx}r_j) - 4\lambda_jq_jr_j - 2\lambda_j(q_jr_j)_{xx}), \tag{2.18a}$$

$$b = \sum_{j=1}^{n}-\frac{1}{2}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(\lambda(q_jr_{j,x} - q_{j,x}r_j) + \lambda_j(q_jr_j)_x), \tag{2.18b}$$

$$c = \sum_{j=1}^{n}\frac{\lambda\lambda_j^3}{\lambda_j^2-\lambda^2}q_jr_j. \tag{2.18c}$$

In this way, we obtain the Lax pair for (2.10*a*) under (2.10*b*) and (2.10*c*) as follows,

$$\psi_{xxx} = \psi_x - m\lambda\psi, \tag{2.19a}$$

$$\psi_t = -\frac{1}{\lambda}\psi_{xx} - u\psi_x + \left(u_x + \frac{2}{3\lambda}\right)\psi$$
$$+ \left(\sum_{j=1}^{n} \frac{1}{6}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(3\lambda(q_j r_{j,xx} - q_{j,xx} r_j) - 4\lambda_j q_j r_j - 2\lambda_j (q_j r_j)_{xx})\right)\psi$$
$$+ \left(\sum_{j=1}^{n} -\frac{1}{2}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(\lambda(q_j r_{j,x} - q_{j,x} r_j) + \lambda_j (q_j r_j)_x)\right)\psi_x$$
$$+ \left(\sum_{j=1}^{n} \frac{\lambda\lambda_j^3}{\lambda_j^2-\lambda^2} q_j r_j\right)\psi_{xx}, \tag{2.19b}$$

which means that the DPESCS is Lax integrable.

3. The infinite set of conservation laws for the DPESCS

With the help of the Lax representation of the DPESCS, we could find the conservation laws for the DPESCS by a well-known method. First we assume that m, u and its derivatives tend to 0 when $|x| \to \infty$, and assume that q_j, r_j and its derivatives tend to 0 when $x \to -\infty$. Set

$$\Gamma = \frac{\psi_x}{\psi}, \tag{3.1}$$

then the identity

$$\frac{\partial}{\partial t}\left(\frac{\partial \ln \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \ln \psi}{\partial t}\right)$$

together with (2.11) implies that DPESCS has the following conservation law:

$$\frac{\partial}{\partial t}(\Gamma) = \frac{\partial}{\partial x}\left(\frac{\psi_t}{\psi}\right) = \frac{\partial}{\partial x}(u_x + a - (u+b)\Gamma - \left(\frac{1}{\lambda} + c\right)(\Gamma_x + \Gamma^2)), \tag{3.2}$$

where a, b and c are given by (2.14). Here we define that $\Omega = u_x + a - (u+b)\Gamma - \left(\frac{1}{\lambda} + c\right)(\Gamma_x + \Gamma^2)$. Using (2.11$a$) gives rise to

$$\Gamma - \Gamma_{xx} = m\lambda + 3\Gamma\Gamma_x + \Gamma^3. \tag{3.3}$$

We can have two kinds of expansions for Γ in the power series of λ. The first expansion is in positive powers of λ [22],

$$\Gamma = \sum_{k=0}^{\infty} h_k \lambda^{k+1}, \tag{3.4a}$$

$$\Omega = \sum_{k=0}^{\infty} g_k \lambda^{k+1}. \tag{3.4b}$$

We note that the odd densities h_{2k+1} are exact derivatives. The first two densities are $h_0 = u, h_2 = u^3$, which yield the conserved quantities H_0 and H_1 in (2.7) and (2.8). The following densities are nonlocal because that would be an inverse of the operator $\left(1 - \partial_x^2\right)$ in the sequence.

The second expansion could be

$$\Gamma = \sum_{k=0}^{\infty} \mu_k \lambda^{\frac{1-k}{3}}, \tag{3.5a}$$

$$\Omega = \sum_{k=0}^{\infty} \omega_k \lambda^{\frac{1-k}{3}}. \tag{3.5b}$$

With the relations of (3.2) and (3.3), we can obtain the infinite densities and fluxes of the conservation laws. For brevity, we omit the recursion relations here.

After some calculations, we can find that μ_k with odd subscripts are derivatives of some functions. So we define $H_{-s} = \int \mu_{2s-2}\,\mathrm{d}x$, which are taken as the conserved quantities. We could find the first few conserved quantities given by μ_0, μ_2 as follows:

$$H_{-1} = \int m^{1/3}\,\mathrm{d}x, \tag{3.6a}$$

$$H_{-2} = \frac{1}{27}\int \left(m_x^2 m^{-7/3} + 9m^{-1/3}\right)\mathrm{d}x. \tag{3.6b}$$

The corresponding flux of the conservation laws are

$$G_{-1} = m^{1/3}\left(-u + \frac{1}{2}\sum_{j=1}^{n}\lambda_j^2(q_j r_{j,x} - q_{j,x} r_j)\right), \tag{3.7a}$$

$$G_{-2} = \frac{1}{2}u_x\sum_{j=1}^{n}\lambda_j^2(q_j r_{j,xx} - q_{j,xx} r_j)) - \frac{1}{3}m_x m^{-1}\left(-u + \frac{1}{2}\sum_{j=1}^{n}\lambda_j^2(q_j r_{j,x} - q_{j,x} r_j)\right). \tag{3.7b}$$

As the space part of the Lax pair is the same as that of DP equation, the densities of the conservation laws are also the same. Of course, as the time part is different, the fluxes will also be different.

4. Solution of the DPESCS

4.1. One peakon solution of the DPESCS

As mentioned in the introduction, we will construct peakon solutions for DPESCS by the method of variation of constants.

The DP equation has peakon solutions [22]. The one peakon is

$$u = c\,\mathrm{e}^{-|x-ct+\alpha|}, \tag{4.1}$$

where α is an arbitrary constant. The corresponding eigenfunction of (4.1) is

$$q = \beta[\mathrm{sgn}(x - ct + \alpha)\,(\mathrm{e}^{-|x-ct+\alpha|} - 1) - 1], \tag{4.2a}$$

$$r = \beta[\mathrm{sgn}(x - ct + \alpha)\,(\mathrm{e}^{-|x-ct+\alpha|} - 1) + 1], \tag{4.2b}$$

where β is arbitrary constant as well.

Taking α and β in (4.1) and (4.2) to be time-dependent $\alpha(t)$ and $\beta(t)$ and requiring that

$$u = c\,\mathrm{e}^{-|x-ct+\alpha(t)|}, \qquad (4.3a)$$

$$q = \beta(t)[\mathrm{sgn}(x-ct+\alpha(t))\,(\mathrm{e}^{-|x-ct+\alpha(t)|}-1)-1], \qquad (4.3b)$$

$$r = \beta(t)[\mathrm{sgn}(x-ct+\alpha(t))\,(\mathrm{e}^{-|x-ct+\alpha(t)|}-1)+1] \qquad (4.3c)$$

satisfies the DPESCS (2.10) for $n=1$. We find that $c=\frac{1}{\lambda_1}$, $\alpha(t)$ can be an arbitrary function of t and $\beta(t)=\sqrt{\alpha'(t)c}$. So we have the one peakon solution for (2.9) with $n=1$, $\lambda_1=\lambda=\frac{1}{c}$

$$u = c\,\mathrm{e}^{-|x-ct+\alpha(t)|}, \qquad (4.4a)$$

$$q = c\sqrt{\alpha'(t)}[\mathrm{sgn}(x-ct+\alpha(t))\,(\mathrm{e}^{-|x-ct+\alpha(t)|}-1)-1], \qquad (4.4b)$$

$$r = c\sqrt{\alpha'(t)}[\mathrm{sgn}(x-ct+\alpha(t))\,(\mathrm{e}^{-|x-ct+\alpha(t)|}-1)+1]. \qquad (4.4c)$$

The one peakon of the DPESCS also has a cusp at its peak, located at $x=ct-\alpha(t)$. We note that, while for the one peakon solution of the DP equation travels with speed c and has a cusp at its peak of height c, for the DPESCS, the cusp is still at its peak of height c, but the speed of the wave is no longer a constant.

4.2. The peakon–antipeakon solution of the DPESCS

The DP equation has peakon–antipeakon solutions [22]

$$u = \frac{-\mathrm{sgn}(t)c}{1-\mathrm{e}^{-2c|t|}}\,(\mathrm{e}^{-|x+c|t|+\alpha|}-\mathrm{e}^{-|x-c|t|+\alpha|}), \qquad (4.5)$$

where α is an arbitrary constant. The corresponding eigenfunctions of (4.5) is

$$q = \beta\sqrt{\frac{c}{(1-\mathrm{e}^{-2c|t|})}}(\mathrm{sgn}(x+c|t|+\alpha)\,(\mathrm{e}^{-|x+c|t|+\alpha|}-1) - \mathrm{sgn}(x-c|t|+\alpha)\,(\mathrm{e}^{-|x-c|t|+\alpha|}-1)), \qquad (4.6a)$$

$$r = \beta\sqrt{\frac{c}{(1-\mathrm{e}^{-2c|t|})}}(\mathrm{sgn}(x+c|t|+\alpha)\,(\mathrm{e}^{-|x+c|t|+\alpha|}-1) + \mathrm{sgn}(x-c|t|+\alpha)\,(\mathrm{e}^{-|x-c|t|+\alpha|}-1)). \qquad (4.6b)$$

Taking α and β in (4.5) and (4.6) to be time-dependent $\alpha(t)$ and $\beta(t)$, and with the method of the variation of constants, the peakon–antipeakon solution of the DPESCS with $n=1$ and $\lambda_1=\frac{1}{c}$ is

$$u = \frac{c}{1-\mathrm{e}^{-2c|t|}}\,(\mathrm{e}^{-|x+c|t|+\alpha(t)|}-\mathrm{e}^{-|x-c|t|+\alpha(t)|}), \qquad (4.7a)$$

$$q = c\sqrt{\frac{-\mathrm{sgn}(t)\alpha'(t)}{(1-\mathrm{e}^{-2c|t|})}}(\mathrm{sgn}(x+c|t|+\alpha(t))\,(\mathrm{e}^{-|x+c|t|+\alpha(t)|}-1) - \mathrm{sgn}(x-c|t|+\alpha(t))\,(\mathrm{e}^{-|x-c|t|+\alpha(t)|}-1)), \qquad (4.7b)$$

$$r = c\sqrt{\frac{-\mathrm{sgn}(t)\alpha'(t)}{(1-\mathrm{e}^{-2c|t|})}}(\mathrm{sgn}(x+c|t|+\alpha(t))\,(\mathrm{e}^{-|x+c|t|+\alpha(t)|}-1) + \mathrm{sgn}(x-c|t|+\alpha(t))\,(\mathrm{e}^{-|x-c|t|+\alpha(t)|}-1)), \qquad (4.7c)$$

where $\alpha(t)$ is an arbitrary function of t.

5. Conclusion

As a concluding remark, we want to stress that the construction of N-peakon solutions for the DPESCS is still an open problem, as well as for the case of the Camassa–Holm equation with self-consistent sources. In fact, in the case of the Camassa–Holm equation and DP equation a reciprocal transformation is used, but so far we have not been able to extend it in the case of those equations with self-consistent sources.

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